Dimension of measures. Egglestón's theorem. Dimension of an ergodic measure for a holomorphic dynamical system.

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Billingsley's lemma leads to the notion of

Def Dimension of a measure: μ - Borel measure in $\mathbb{R}^d$.

$\dim \mu = \inf \{ \mathrm{Hdim}\, A : \mu(A^c) = 0,\ A \subset \mathbb{R}^n - \text{Borel} \}$

Another, equivalent, def

$\dim_\mu = \inf \{ \alpha : \mu \perp m_\alpha \}$.

Remark. inf = min, take A_n with $\mu(A_n^c) = 0$, $\mathrm{Hdim}\, A_n < \dim_\mu + \frac{1}{n}$). $A := \cap A_n$.

Local dimension of μ at x is defined as

$\dim_\mu^b(x) = \varliminf_{n\to\infty} \frac{\log \mu(Q_n(x))}{-n \log b}$.

Remark This definition, strictly speaking, depends on b.

An independent definition: $\dim_\mu(x) = \varliminf_{r\to 0} \frac{\log \mu(B(x,r))}{|\log r|}$. Since $\mu(B(x,r)) \geq \mu(Q_n(x))$ for some n with $b^{-n} \simeq r$, we always have $\dim_\mu(x) \leq \dim_\mu^b(x)$.

An example of μ be uniform on $[0, \frac{1}{2}]$, shows that at $\frac{1}{2}$, $\dim_\mu^2 = \infty$, $\dim_\mu = 1$.

Equivalent b-adic definition: consider $\max_{x \in \bar{Q}} \mu(Q)$.

Also easy to see, using a version of Billingsley Lemma, that $\mu\{x : \dim_\mu^b(x) \neq \dim_\mu(x) \text{ for some } b\} = 0$.

We'll work with $\dim_\mu^b$, to simplify the derivations.

Lemma $\dim \mu = \operatorname{ess\,sup} \dim_\mu^b(x)$ ($\operatorname{ess\,sup} f := \inf\{M : \mu\{f(x) \geq M\} = 0\}$).

Pf. Pick $\alpha > \operatorname{ess\,sup} \dim_\mu(x)$, let

$A := \{ x : \varliminf_{n\to\infty} \frac{\log \mu(Q_n(x))}{-n \log b} \leq \alpha \}$. $\mu(A^c) = 0$, and,

by Billingsley, $\mathrm{Hdim}\, A \leq \alpha$

If $\alpha < \operatorname{ess\,sup}$, $\not\exists$ $B := \{ x : \varliminf_{n\to\infty} \frac{\log \mu(Q_n(x))}{-n \log b} \geq \alpha \}$.

$\mu(B) > 0$. If $\mu(E^c) = 0$, then

$\mu(B) = \mu(E \cap B) > 0$ and $\varliminf \geq \alpha$ on $E \cap B$.

By Billingsley, $\mathrm{Hdim}\, E \geq \mathrm{Hdim}\, E \cap B \geq \alpha$

Remark. $\underline{\dim}\, \mu := \inf \{ \mathrm{Hdim}(A) : \mu(A) > 0 \} = \operatorname{ess\,inf} \dim_\mu^b(x)$ - with the same proof.

Slightly different: $\overline{\dim}_\mu^b(x) := \varlimsup_{n\to\infty} \frac{\log \mu(Q_n(x))}{-n \log b}$. Then $\mathrm{Pdim}\, \mu$ and $\underline{\mathrm{Pdim}}\, \mu$, defined as $\inf\{\mathrm{Pdim}\, A : \mu(A^c) = 0\}$ and $\inf\{\mathrm{Pdim}\, A : \mu(A) = 0\}$ are equal to $\operatorname{ess\,sup} \overline{\dim}_\mu^b(x)$ and $\operatorname{ess\,inf} \overline{\dim}_\mu^b(x)$.

An application, and a first example of multifractal analysis: Eggleston's thm.

Let us remind that $T_2(x) = 2x \bmod 1 : [0,1] \to [0,1]$.

If $x = \sum x_n 2^{-n}$, then $x_k = \chi_{[\frac{1}{2},1]}(2^k x \bmod 1) = \chi_{[\frac{1}{2},1]}(T_2^{\circ k}(x))$.

T_2 preserves Lebesgue measure, $|T_2^{-1}(B)| = |B|$), and

is) ergodic, i.e. if B is T_2 invariant ($T_2^{-1}(B) = B$), then $|B| = 1$ or 0.

Then, by the famous Birkhoff ergodic theorem, for any $f \in L^1$,

$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^{\circ k} x) \to \int_0^1 f(t)\,dt$ μ-a.s.

In particular, for $f = \chi_{[\frac{1}{2},1]}$ $\frac{\sum_{k=1}^n x_k}{n} \to \frac{1}{2}$ a.s by μ.

Let us look at an exceptional set

$A_p := \{ x : \frac{\sum x_k}{n} \to p \}$ $p \neq \frac{1}{2}$, $0 \leq p \leq 1$, and

try to compute its dimension.

To this end, define measure μ_p. It is enough to define it for the binary intervals:

$\mu_p([j 2^{-n}, (j+1) 2^{-n})) = p^{k(j)} (1-p)^{n-k(j)}$, where

$k(j)$ is the number of 1s in the binary expansion of j

Alternatively, define inductively by assigning probability p to the right interval, $1-p$ - to the left: I.e. $\mu_p(Q_n(x)) = p^{\sum_{k=1}^n x_k} (1-p)^{\sum_{k=1}^n (1-x_k)}$.

$1-p$ p

$p = \frac{1}{2}$ - Lebesgue measure.

Define $h_2(p) := -p \log_2 p - (1-p) \log_2 (1-p)$ the entropy of μ_p with respect to T_2. I will return to the meaning of this later, but for now let us prove that

Lemma $\mathrm{Hdim}\, A_p = \dim \mu_p = h_2(p)$.

Proof. First, let us observe that μ_p is also T_2-invariant and ergodic.

Indeed let E be a T_2-invariant set. A - a dyadic set with $\mu_p(E \Delta A) < \varepsilon$, $A = \cup T\ldots$

[illegible] let us prove that

Lemma $\mathrm{Hdim}\, A_p = \dim \mu_p = h_2(p)$.

Proof. First, let us observe that μ_p is also T_2-invariant and ergodic.
Indeed, let E be a T_2-invariant set, A - a dyadic set with $\mu_p(E \Delta A) < \varepsilon$, $A = \bigcup_{\substack{\sigma\in\Sigma \\ \text{length } \sigma = n}} I_{\sigma}$.
Then $T^{-n}A \cap A = \bigcup_{\sigma_1,\sigma_2 \in \Sigma} I_{\sigma_1\sigma_2}$, so $\mu_p(T^{-n}A \cap A) = \mu_p(A)^2$.
But $E \Delta T^{-n}A = T^{-n}E \,\Delta\, T^{-n}A = T^{-n}(E\Delta A)$, so
$|\mu_p(E) - \mu_p(E)^2| \le |\mu_p(E) - \mu_p(T^{-n}A \cap A)| + |\underbrace{\mu_p(T^{-n}A\cap A)}_{=\mu_p(A)^2} - \mu_p(E)^2| \le$
$2\varepsilon + |\mu_p(E) - \mu_p(T^{-n}A)| + |\mu_p(E) - \mu_p(A)| \le 4\varepsilon$. So
$|\mu_p(E) - \mu_p(E)^2| \le 4\varepsilon \;\forall \varepsilon > 0$, so $m(E) \in \{0,1\}$. Same way, if $\vec{p} = (p_1, \dots, p_b)$ - probability vector, then $\mu_{\vec{p}}$ is T_b-ergodic.
Thus μ_p-a.s. $\frac{1}{n}\sum x_k \to \int_0^1 \chi_{[\frac{1}{2},1]}(t)\, d\mu_p(t) = p$ (Another way to see it is by SLLN. x_k are i.i.d. $\{0,1\}$ with $(1-p, p)$).
Thus $\mu_p(A_p) = 1$.
$\dim \mu_p = \operatorname{ess\,sup} \dim_\mu(x)$. But

$$\frac{\log \mu_p(Q_n(x))}{-n\log 2} = \frac{1}{\log 2}\left(\left(\frac{1}{n}\sum_{k=1}^n x_k\right)\log\frac{1}{p} + \left(\frac{1}{n}\sum_{k=1}^n (1-x_k)\right)\log\frac{1}{1-p}\right).$$

So, as $n\to\infty$, since μ-a.e. $x \in A_p$, it converges to

$$\frac{p\log\frac{1}{p} + (1-p)\log\frac{1}{1-p}}{\log 2} = h_2(p).$$

So $h_2(p) = \dim \mu_p \le \dim A_p$. But for any $x \in A_p$,
$\dim_{\mu_p}(x) = h_2(p)$, by above. So, by Billingsley,
$\dim A_p \le h_p$, since $\mu_p(A_p) > 0$. ∎ Observe that $\dim_{\mu_p} x$ exists μ_p-a.e. as a limit, not $\underline{\lim}$.

Remark. A generalization:
If $\vec{p} = (p_0, \dots, p_{b-1})$ - b-multi-index, with
$p_0 + \dots + p_{b-1} = 1$, $p_j \ge 0$. Then define
$\mu_{\vec{p}}(Q_n(x)) := \prod_{j=1}^n p_{x_j}$, $x = \{x_j\}$ - b-ary expansion.
$A_{\vec{p}} := \{x : \frac{\#\{k \le n,\, x_k = j\}}{n} \to p_j,\ j = 0, \dots, b-1\}$
Then, exactly as above,
$\dim \mu_{\vec{p}} = \mathrm{Hdim}\, A_{\vec{p}} = h_b(\vec{p}) = -\sum_{j=0}^{b-1} p_j \log p_j$

What is the dynamic meaning of the entropy? Consider again $b=2$.
For x, consider $\frac{\mu(T_2(B(x,r)))}{\mu(B(x,r))}$ for small r, or,
equivalently, $\frac{\mu_p(T_2(Q_n(x)))}{\mu_p(Q_n(x))}$. But $T_2(Q_n(x)) = Q_{n-1}(T_2 x)$, so the measure is divided by p (if $x_1 = 1$) or $(1-p)$ (if $x_1 = 0$).
Let us define Jacobian of μ $J_\mu := \lim_{n\to\infty} \frac{\mu_p(T_2(Q_n(x)))}{\mu_p(Q_n(x))} = \begin{cases} \frac{1}{p}, & \text{if } x_1 = 1 \\ \frac{1}{1-p}, & \text{if } x_1 = 0 \end{cases}$
One can compute $\mu_p(Q_n(x)) \approx J_\mu(x) \cdot J_\mu(T_2 x) \dots J_\mu(T_2^{\circ n-1} x)\, \mu_p[0,1]$, so

$$\frac{\log \mu_p(Q_n(x))}{n} \approx \frac{1}{n}\sum \log(J_{\mu_p}(T_2^{\circ k} x)) \to \int \log J_{\mu_p}\, d\mu_p \text{ - ergodic theorem!}$$

$= p\log\frac{1}{p} + (1-p)\log\frac{1}{1-p}$ - average expansion of μ. Notation: entropy of μ, $h_\mu(T) = h_\mu$.
On the other hand, average log expansion of distance is $\log 2$ - the map is linear, so it is easy. But let us rewrite it as $\lambda(\mu, T) = \int \log|T_2'|\, d\mu_p = \log 2$ - the Lyapunov exponent of μ.
By Ergodic Theorem, $\lambda(\mu) = \frac{1}{n}\sum_{k=0}^{n-1} \log T_2'(T_2^{\circ k}(x))$ for μ_p-a.e. x.

So, heuristically, after one iteration $\log\mu$ increases by h_μ, and the radius increases by $\lambda(\mu)$.
So, again heuristically, we get the right result here: $\dim_\mu(x) = \frac{h_\mu}{\lambda_\mu}$ μ-a.e., and, consequently, $\dim\mu = \frac{h_\mu}{\lambda_\mu}$.
Turns out, that it has far-reaching generalizations. Let me state a 1D version.

Thm (Mañé - Przytycki) (Volume lemma).
Let $X \subset \mathbb{C}$ - compact, $f : X \to X$ - analytic in some nbhd of X, μ - f-invariant ergodic measure on X, $\lambda(\mu) > 0$ (so, on μ-average, f is exponentially expanding).
Then μ-a.e. $\dim_\mu(x) = \frac{h_\mu}{\lambda_\mu} = \lim_{r\to 0} \frac{\log\mu(B(x,r))}{|\log r|}$ (exists as a limit!)